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The accuracy and parametric sensitivity of algebraic models for turbulent flow and convection

Stuart W. Churchill [a,*], Bo Yu [b], Yasuo Kawaguchi [c]

[a] *Department of Chemical and Biomolecular Engineering, University of Pennsylvania, 311A Towne Building, 220 South 33rd Street, Philadelphia, PA 19104, USA*
[b] *Oil and Gas Storage and Transport Engineering, China University of Petroleum, Beijing 102249, People's Republic of China*
[c] *Department of Mechanical Engineering, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamakazi, Noda, Chiba 278-8510, Japan*

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Abstract

Predictions of turbulent flow and convection in round tubes, channels and concentric circular annuli based on a correlating equation for the fraction of the shear stress due to turbulence have previously been shown to be within the scatter of the best experimental data. However, the sufficiency of this agreement has been questioned because of the scatter and limitations in scope of that data. As a supplementary test, the sensitivity of the predictions to each of the numerical empiricisms and arbitrary functions of the model has been investigated. On the whole, the uncertainties in these values and functions are not found to influence the predictions significantly.

1. Introduction

Churchill and coworkers utilized the fraction of the local shear stress due to turbulence as a variable in the time-averaged differential momentum balance and thereby avoided the imposition of a heuristic quantity such as the eddy diffusivity. They then devised an algebraic model for this quantity in terms of a power-mean of theoretical asymptotes with a mix of theoretical and empirical coefficients and an empirical combining exponent. This algebraic model, which incorporates less empiricism than any prior model, including those of κ–ε and *LES*, was used to compute the velocity distribution and mixed-mean rate of flow in round tubes, parallel-plate channels, and concentric circular annuli. The latter geometry invokes a slight additional empiricism in the form of correlating equations for the location of the maximum in the velocity and the zero in the total shear stress.

The predictions by this model of the time-averaged velocity and the mixed-mean velocity, which is equivalent to the friction factor, were found to agree well with the best experimental data for all three geometries. For round tubes, the predictions are in almost perfect agreement with the very precise experimental data of Zagarola [18] but that agreement is to some degree forced in that two of the empirical constants of the model were based on these very data. The good agreement that was found for the predictions with the experimental data for parallel-plate channels and annuli is free of such forcing but is at the same time less convincing because of the limited scope, the great scatter, and the obvious

* Corresponding author. Tel.: +1 215 898 5579; fax: +1 215 573 2093.
E-mail address: churchil@seas.upenn.edu (S.W. Churchill).

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Nomenclature

a	radius (m)
a_1	inner radius of annulus (m)
a_2	outer radius of annulus (m)
a_0	radius of zero shear stress in an annulus (m)
a_{max}	radius of maximum in velocity in an annulus (m)
a^+	dimensionless radius $[a(\tau_w\rho)^{1/2}/\mu]$
A	arbitrary constant in semi-logarithmic expression for velocity
b^+	dimensionless half-width $[b(\tau_w\rho)^{1/2}/\mu]$
b	half-width of parallel-plate channel (m)
c	specific heat capacity (J/kg K)
f	Fanning friction factor $[2\tau_w/\rho u_m^2]$
j	radial heat flux density (W/m^2)
j_w	radial heat flux density at wall (W/m^2)
k	coefficient in semi-logarithmic expression for velocity
ℓ	mixing length (m)
ℓ^+	dimensionless mixing length $[\ell(\tau_w\rho)^{1/2}/\mu]$
n	arbitrary combining exponent
Nu	Nusselt number $[2h(a_2 - a_1)/\lambda]$
Pr	Prandtl number $[c\mu/\lambda]$
Pr_t	turbulent Prandtl number $[c\mu_t/\lambda_t]$
Pr_T	total Prandtl number $[c(\mu + \mu_t)/(\lambda + \lambda_t)]$
R	radius ratio $[1 - (y/a)]$
Re	Reynolds number $[2(a_1 - a_2)u_m/\mu]$
T	time-averaged temperature (K)
T^+	dimensionless temperature $[\lambda(\rho\tau_w)^{1/2}(T_w - T)/\mu j_w]$
T_m	mixed-mean temperature (K)
T'	fluctuating component of temperature (K)
$\overline{T'v'}$	time-average of product of fluctuating temperature and velocity (K m/s)
$(\overline{T'v'})^{++}$	local fractional of radial heat flux density due to turbulence $[\rho c\overline{T'v'}/j]$
u	axial component of time-averaged velocity (m/s)
u^+	dimensionless axial velocity $[u(\rho/\tau_w)^{1/2}]$
u_m	mixed-mean axial velocity (m/s)
u'	fluctuating component of axial velocity (m/s)
$\overline{u'v'}$	time-average of product of fluctuating components of velocity (m^2/s^2)
$(\overline{u'v'})^{++}$	local fraction of shear stress due to turbulence $[-\rho\overline{u'v'}/\tau]$
$\overline{(u'v')}^+$	dimensionless shear stress $[-\rho\overline{u'v'}/\tau_w]$
v	component of time-averaged velocity normal to wall (m/s)
v'	fluctuating component of velocity normal to the wall (m/s)
y	distance from wall (m)
y^+	dimensionless distance from wall $[y(\rho\tau_w)^{1/2}/\mu]$
Z	$1 - (y/b)$

Greek symbols

α	arbitrary coefficient
γ	$[(j/j_{w1})(\tau_{w1}/\tau) - 1]$
ε	rate of dissipation of turbulence (m^2/s^3)
κ	kinetic energy of turbulence (m^2/s^2)
λ	thermal conductivity (W/m K)
μ	dynamic viscosity (Pa s)
μ_t	eddy dynamic viscosity (Pa s)
ξ	$[1 - (\overline{T'v'})^{++}]/[1 - (\overline{u'v'})^{++}]$
ρ	specific density (kg/m^3)
τ	shear stress (Pa)
τ_w	shear stress at wall (Pa)

discrepancies between different sets of data, particularly for annuli. Because of the use of irregular values of the added parameter, namely the aspect ratio, by different experimenters, the comparisons for annuli were made with the predictions of correlating equations for the computed values rather than directly with the computed values, which were for regular values of the aspect ratio. This interposition provides a more severe rather than a less critical test.

This approach was extended to forced convection by utilizing the local heat flux density due to turbulence as a variable in the time-averaged differential energy balance. The thermal modeling encompassed round tubes with both uniform and isothermal heating, and parallel-plate channels and annuli with all combinations of uniform and isothermal heating and cooling on the two surfaces. The only empiricism beyond that of the correlating equation for turbulent shear stress is that of the correlating equation for the turbulent Prandtl number, which bears a one-to-one correspondence to the turbulent heat flux density. All in all, the algebraic modeling for turbulent convection incorporates less empiricism than that of any prior modeling.

The predictions of the Nusselt number by the algebraic thermal model were found to be in better agreement with the experimental data for all three geometries, all values of the Prandtl number, and all modes of heating than any prior ones, even those for a single geometry and condition. Because of the irregular values of the two thermal parameters in the various sets of experiments, the comparisons were with correlating equations for the computed values rather than directly. Just as with the aspect ratio of annuli, this interposition increases rather than decreases the criticality of the comparisons. The agreement, although very good overall, is not totally convincing because of the even greater scatter, the greater differences between various sets, and the lesser scope of the data.

The inability to evaluate the numerical uncertainties of the new modeling by comparisons with experimental data because of their limitations in precision, accuracy, and scope led to the analysis presented herein, which has the objective of identifying each empirical or arbitrary element in the modeling and of assessing the associated quantitative uncertainty of the predictions. This analysis has three aspects. The first aspect consists of a detailed description and critique of the algebraic expressions for the turbulent shear stress and the turbulent heat flux density that are inherent in the modeling of turbulent flow and convection. These expressions have been described and critiqued previously, but only in segments as they evolved, not as a whole. The second aspect consists of the identification of the individual empiricisms and arbitrary functional elements of the fluid-mechanical and thermal models, including the particular ones for each separate geometry. The third aspect consists of the numerical evaluation of the sensitivity of the numerical prediction of turbulent flow and convection to each of these empiricisms and arbitrary elements.

The questions that this analysis is intended to answer include the following. Is the agreement of the predictions of the new modeling with experimental data a fortuitous result of their scatter? Are there any aspects of the modeling that are not theoretically sound? How sensitive are the predictions of the new models to the numerical values of their coefficients and exponents as well as to the functionality of the various elements of their structure? Will future experimental data or the results of *DNS* for extended conditions require the tweaking of the empirical coefficients and/or require modification of their functional forms? These questions are not rhetorical. Their answers may not only characterize the reliability of the present predictive models, but may also identify the nature and ranges of the future experiments and/or calculations that are most critical and essential.

2. Prior models

Most models for the prediction of fully developed turbulent flow and forced convection in channels are based on the time-averaged equations of conservation. The time-averaging greatly simplifies these equations but at the expense of generating additional unknown dependent variables such as $\overline{u'v'}$ and $\overline{T'v'}$. Most of the modeling of turbulent flow that starts from the time-averaged equation for the conservation of momentum involves the introduction of heuristic quantities such as the eddy viscosity of Boussinesq [1] and the mixing-length of Prandtl [2] for $\overline{u'v'}$. For turbulent flow in channels, both empirical expressions and arbitrary predictive models have in turn been proposed for the eddy viscosity and the mixing length. Early examples of predictive models are those of Deissler [3,4] and of von Kármán [5]. Later predictive models for the eddy viscosity are epitomized by κ–ε model of Kolmogorov [6], Prandtl [7], and Batchelor [8]. The kinetic energy of turbulence $\kappa$and rate of dissipation of turbulence εin this model have physical identities, but the equations devised by Launder and Spalding [9] and others for their prediction are arbitrary. The κ–ε-$\overline{u'v'}$ model of Hanjalić and Launder [10] avoids the introduction of a heuristic quantity but requires arbitrary predictive equations for each of these three quantities. The corresponding models for turbulent convection have generally consisted of simple modifications of those for flow.

When *DNS* (direct numerical simulation), which avoids time-averaging and is virtually free of empiricism, was introduced some 20 years ago it seemed to have unlimited promise. However, its applications for both flow and convection are still essentially limited to planar flows and to rates of flow barely above the lower limit for fully developed turbulence. *LES* (large-eddy simulation) relaxes the restriction on the rate of flow by utilizing *DNS* only for the fully turbulent core, but requires the use of the κ–ε model with arbitrary wall functions or the equivalent for the region near the wall.

3. The new model for flow

3.1. Basic formulations for a round tube

The new modeling is based on the introduction by Churchill [11] of the following dimensionless variable to represent the radial transport of momentum by the turbulent fluctuations:

$$(\overline{u'v'})^{++} = \frac{-\rho(\overline{u'v'})}{\tau} \tag{1}$$

Here, τ is the local total shear stress, and $(\overline{u'v'})^{++}$ may be interpreted as the fraction of the shear stress due to the turbulent fluctuations. The time-averaged equation for the conservation of momentum in the fully developed flow of a fluid with invariant viscosity and density in a round tube may be expressed in terms of this new variable as follows:

$$[1-(y^+/a^+)][1-(\overline{u'v'})^{++}] = \frac{du^+}{dy^+} \tag{2}$$

Here, u, y, and a are scaled in terms of conventional wall-variables, that is as $u^+ \equiv u(\rho/\tau_w)^{1/2}$, $y^+ \equiv y(\rho\tau_w)^{1/2}/\mu$, and $a^+ \equiv y(\rho\tau_w)^{1/2}/\mu$. Eq. (2) may be re-expressed in terms of $R = 1-(y^+/a^+)$ and integrated formally from $u^+ = 0$ at the wall ($R = 1$) to obtain

$$u^+ = \frac{a^+}{2}\int_{R^2}^{1}[1-(\overline{u'v'})^{++}]\,dR^2 \tag{3}$$

The local, time-averaged, dimensionless velocity, u^+, as given by Eq. (3), may be integrated by parts over the circular cross-section to obtain the following single integral for the dimensionless mixed-mean velocity, $u_m^+ \equiv u_m(\rho/\tau_w)^{1/2}$, and its counterpart, the Fanning friction factor, $f = 2\tau_w/\rho u_m^2$:

$$\left(\frac{2}{f}\right)^{1/2} \equiv u_m^+ \equiv \int_0^1 u\,dR^2 = \frac{a^+}{4}\int_0^1 [1 - (\overline{u'v'})^{++}]\,dR^4 \quad (4)$$

Eqs. (2)–(4) are exact, but empiricism is introduced unavoidably by the correlative equation that is utilized for $(\overline{u'v'})^{++}$.

3.2. An algebraic model for the turbulent shear stress in a round tube

The original correlative equation devised by Churchill and Chan [12] for the turbulent shear stress in the dimensionless form of $(\overline{u'v'})^+ \equiv -\rho(\overline{u'v'})/\tau_w = (\overline{u'v'})^{++}/[1 - (y^+/a^+)]$ has evolved on the basis of improved experimental data to the following one in terms of the new dimensionless variable of Churchill [11]:

$$(\overline{u'v'})^{++} = \left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7} + \left|\exp\left\{\frac{-1}{0.436y^+}\right\} - \frac{1}{0.436a^+}\left(1 + \frac{6.95y^+}{a^+}\right)\right|^{-8/7}\right)^{-7/8} \quad (5)$$

Eq. (5) has the form of the generalized correlating equation of Churchill and Usagi [13], namely the power-mean of two or more asymptotes or limiting solutions. In this application it is made up of the following three asymptotes:

$$(\overline{u'v'})^{++} = 0.0007(y^+)^3 \quad \text{for } y^+ \to 0 \quad (6)$$

$$(\overline{u'v'})^{++} = 1 - \frac{1}{0.436y^+} \cong e^{-1/0.436y^+} \quad \text{for } 30 \leqslant y^+ \leqslant 0.1a^+ \quad (7)$$

$$(\overline{u'v'})^{++} \to 1 - \frac{20.53}{a^+} \quad \text{for } y^+ \to a^+ \quad (8)$$

The third-power dependence of Eq. (6) was derived some time ago by a number of different investigators, perhaps first by Murphree [14] by means of a speculative asymptotic analysis. For many years, alternative power dependences had a considerable advocacy, as exemplified by the work of Deissler [3,4] who first proposed the equivalent of a second-power dependence and later the equivalent of a fourth-power one. The viscous damping function of van Driest [15], which is often still utilized, also implies a fourth-power dependence. This controversy has now been resolved beyond question in favor of the third power by the results of *DNS*. However, some uncertainty remains with respect to the value of the coefficient due to current limits on the precision of the *DNS* very near the wall.

The un-approximated form of Eq. (7) corresponds to the semi-logarithmic dependence of u^+ on y^+, which was derived by Prandtl [16] using mixing-length theory but later by Millikan [17] by means of speculative dimensional analysis and thereby without introducing a heuristic quantity. The exponential approximation is just a mathematical artifact to avoid singular behavior for $y^+ \leqslant 0.436$ when this asymptote is incorporated in Eq. (5) and thereby evaluated for $y^+ \leqslant 0.436$. The coefficient of 0.436 is based on the recent experimental measurements of the time-averaged velocity by Zagarola [18]. Barenblatt and Prostokushin [19] have asserted that a power of y^+ provides a better representation for the velocity distribution than does a semi-logarithmic expression, but this possible superiority for a narrow range of y^+ is countervailed with respect to its use in Eq. (5) by singular behavior for both large and small values of y^+. The semi-logarithmic regime, which is known on the basis of the idealizations in its derivation as "the turbulent core near the wall", is valid only outside the boundary and "buffer" layers, that is for $y^+ > 30$, and inside the wake, that is for $y^+ < 0.1a^+$. Accordingly, this regime is crowded out of existence for $a^+ < 300$, and Eq. (5) becomes increasingly inaccurate for lesser values of y^+. Because Eq. (5) becomes inapplicability for $a^+ < 150$ by virtue of the onset of laminarization, this source of inaccuracy is limited to $150 \leqslant a^+ \leqslant 300$.

The form of Eq. (8) follows from the proportionality of the velocity defect, $u_c^+ - u^+$, to $[1 - (y^+/a^+)]^2$, a relationship first derived by Hinze [20] on the basis of the irrefutable experimental evidence of the approach of the eddy diffusivity to a finite value as $y^+ \to a^+$. The coefficient 6.95 in Eq. (5) and the coefficient 20.53 in Eq. (8) bear a one-to-one correspondence to each other and to the limiting dimensionless value of the amplitude of the "wake" at the centerline, namely, $u_c^+ - 6.13 - (1/0.436)\ln\{a^+\} = 1.51$, as determined by Zagarola, and were actually determined from that value. Here, 6.13 is the value of the constant in the semi-logarithmic representation of Zagarola for the time-averaged velocity. This value, which does not appear explicitly in Eq. (5), is consistent with the numerical values of u^+ determined by means of Eqs. (3) and (4).

The term inside the absolute-value signs of Eq. (5) was devised (see Churchill and Chan [12]), to encompass the entire region of $30 \leqslant y^+ \leqslant a^+$ for $a^+ > 300$ by interpolating between Eqs. (7) and (8). The absolute value is specified to prevent the singularity of this combined term for $y^+ \to 0$ from persisting in Eq. (5). The $-8/7$-power-mean of that combination and Eq. (6) results in an interpolation between these two expressions in the range of $0 \leqslant y^+ \leqslant 0.1a^+$.

MacLeod [21] speculated on the basis of formulations for laminar flow that $u^+\{y^+,b^+\}$ for parallel-plate channels might be identical to $u^+\{y^+,a^+\}$ for round tubes in the fully turbulent as well as the fully laminar regime. This speculation was confirmed experimentally by Whan and Rothfus [22] for the extreme case of the centerline velocity, but at the same time found to be invalid for the combined regimes of transition. Churchill and Chan [12] inferred from the equivalent of Eq. (1) that the analogy of MacLeod must apply to $(\overline{u'v'})^{++}$ insofar as it does to u^+. Although theoretical considerations suggest that this analogy cannot be exact, it appears nonetheless to be accurate well within the uncertainty of the best experimental data for both u^+ and $(\overline{u'v'})^{++}$. The extended analogy is critical to Eq. (5) in two respects. First, the coefficient of Eq. (6) is based on *DNS* for a parallel-plate channel, and second, the exponent of $-8/7$ was chosen as the value of combining exponent that results in the best overall representation of the experimental data of Wei and Willmarth [23] for $\overline{u'v'}$ in a parallel-plate channel.

The accumulated empiricism inherent in Eq. (5), as described here term by term is effectively less than that involved in the implementation of the κ–ε, κ–ε-$\overline{u'v'}$, and *LES* models.

The evaluation of $u^+\{y^+,a^+\}$ and $u^+_m\{a^+\}$ from Eqs. (3) and (4), respectively, by quadrature or by stepwise integration of their finite-difference equivalents does not introduce any empiricism or arbitrariness beyond that of Eq. (5). Furthermore, the smoothing that is inherent in the integrations by either method has the fortuitous effect of reducing the effects of both the functional and numerical uncertainty of Eq. (5) on the numerically predicted values. These numerically computed values fall entirely within the narrow band of scatter of the experimental data of Zagarola, which is not too surprising because these are the data that served as the source of the coefficients 0.436 and 6.95. On the other hand, the overall agreement is decisively better (see, Churchill, et al. [24]) than that of Zagarola's own correlating equations because of the superiority in form and greater scope of Eq. (5). Also, as implied by the analogy of MacLeod, the numerical values of $u\{y^+,b^+\}$ and $u^+_m\{b^+\}$, as calculated by Danov et al. [25], using Eqs. (3) and (5) and the equivalent of Eq. (4), agree closely with the best experimental data for parallel-plate channels. The success of the predictions also extends to concentric circular annuli (see, Kaneda et al. [26]).

The mention of several precedents was deferred to this point in order to have the above expressions as points of reference. Kampé de Fériet [27] derived the structural equivalent of Eqs. (2) and (3) in terms of $(\overline{u'v'})^+$, as well as their analogues for a parallel-plate channel, but without recognition of the analogy of MacLeod. Pai [28,29] devised separate empirical expressions for $(\overline{u'v'})^+$ for both a parallel-plate and a round tube, which, however, because of singularities, failed when implemented to determine the friction factor. Bird et al. [30] suggested that a correlating equation for the turbulent shear stress might prove simpler than those for the eddy viscosity or the mixing length, but did not proceed any further.

3.3. The relationships between the new and old models

The general relationships, which follow, between the new model and the classical ones, prove to be useful in three senses. First, the classical ones are revealed to have a more fundamental character than might have been expected on the basis of their heuristic origin, second, a fundamental flaw is revealed in one of the classical ones for the first time, and third, a mechanism is provided for the salvage of results that have been determined and expressed in terms of the heuristic variables.

A comparison of the differential momentum balance for a round tube in terms of the eddy viscosity ratio, μ_t/μ, with Eq. (2) reveals that

$$\frac{\mu_t}{\mu} = \frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}} \tag{9}$$

Since $(\overline{u'v'})^{++}$ is positive, finite, and less than unity for all finite values of y^+ and a^+ in the regime of fully developed turbulence, μ_t/μ must be finite and positive for these conditions as well. More importantly, the eddy viscosity is revealed to have the physical sense of the ratio of the shear stress due to the turbulent fluctuations to that due to the molecular motion, and, as such, to be independent of its heuristic diffusional origin. (Boussinesq [1], was either very insightful or lucky.) It further follows that if Eqs. (1)–(8) were re-expressed in terms of the eddy viscosity, they would be equally valid, although more complex algebraically. Finally, all results in the literature expressed in terms of μ_t/μ may be converted quantitatively to values of $(\overline{u'v'})^{++}$ by means of Eq. (9). This correspondence applies to a parallel-plate channel with smooth surfaces but not to annuli, for which the eddy viscosity model is fatally flawed (see, for example, Kaneda et al. [26]).

The equivalent analysis in terms of the mixing length has a quite different outcome. Comparison of the differential momentum balance in terms of the mixing length with Eq. (2) results in a relationship that may be expressed as

$$\left[\left(\frac{\ell}{a}\right)a^+\right]^2 = (\ell^+)^2 = \frac{(\overline{u'v'})^{++}}{\left(1-\frac{y^+}{a^+}\right)\left[1-(\overline{u'v'})^{++}\right]^2} \tag{10}$$

The mixing length is seen from Eq. (10) to have a real, although very complex, physical interpretation in terms of the molecular and turbulent fluctuations, and thereby to be independent of its heuristic origin as an analog of the mean-free-path in a gas. However, in contrast to the eddy viscosity, the mixing length is unbounded at the centerline, a critical aspect of behavior that has been overlooked for 70 years. These discoveries with respect to the mixing length are primarily of historical interest because this variable is no longer widely used, whereas the eddy viscosity still is. As an aside, Nikuradse [31] developed an algebraic correlating equation for the mixing length in terms of asymptotes, thereby anticipating in one sense the concept of Eq. (5), but unfortunately with functionally erroneous asymptotes in both limits.

3.4. The sensitivity of flow in a round tube to the parameters of the algebraic model for the shear stress

The above description identifies the coefficient $\alpha = 0.0007$ of Eq. (6), the coefficient $k = 0.436$ of Eq. (7), and the coefficient $A = 6.95$ and the exponent $n = -8/7$ of Eq. (5) as the only explicit numerical empiricisms. Accordingly, these values are perturbed in the ensuing analysis. The power-mean form, the use of the absolute value, and the combination of the terms inside the absolute value signs of Eq. (5), as well as the exponential approximation of Eq. (7) are also arbitrary, but the consequences of these functional approximations may be shown to be insignificant relative to those of arising from the numerical values of the indicated coefficients and exponent. The unapproximated form of Eq. (7) is itself arbitrary, but the experimental support for the semi-logarithmic dependence of the time-averaged velocity on y^+, upon which this expression is based, is very strong.

The computed values of u^+ for $a^+ = 1000$, 5000, and 50,000 corresponding to $Re = 3.76 \times 10^4$, 2.27×10^5, and 2.80×10^6 are plotted in Fig. 1(a)–(c), respectively. The corresponding values of $u^+_{\rm m}$, as determined by the stepwise simultaneous numerical integration of finite-difference representations of Eq. (2) and the differential equivalent of Eq. (3), respectively, using a standard version of Runge–Kutta, rather than from the integral forms, are listed in Table 1.

3.4.1. The coefficient α

A value of 0.0007, as determined from the values of $(\overline{u'v'})^{++}$ calculated by Rutledge and Sleicher [32] using DNS, was chosen for Eq. (5), and thereby for the base case to which the various perturbations are referred. Test calculations were carried out for $\alpha = 0.00068$ and 0.00072, corresponding to downward and upward

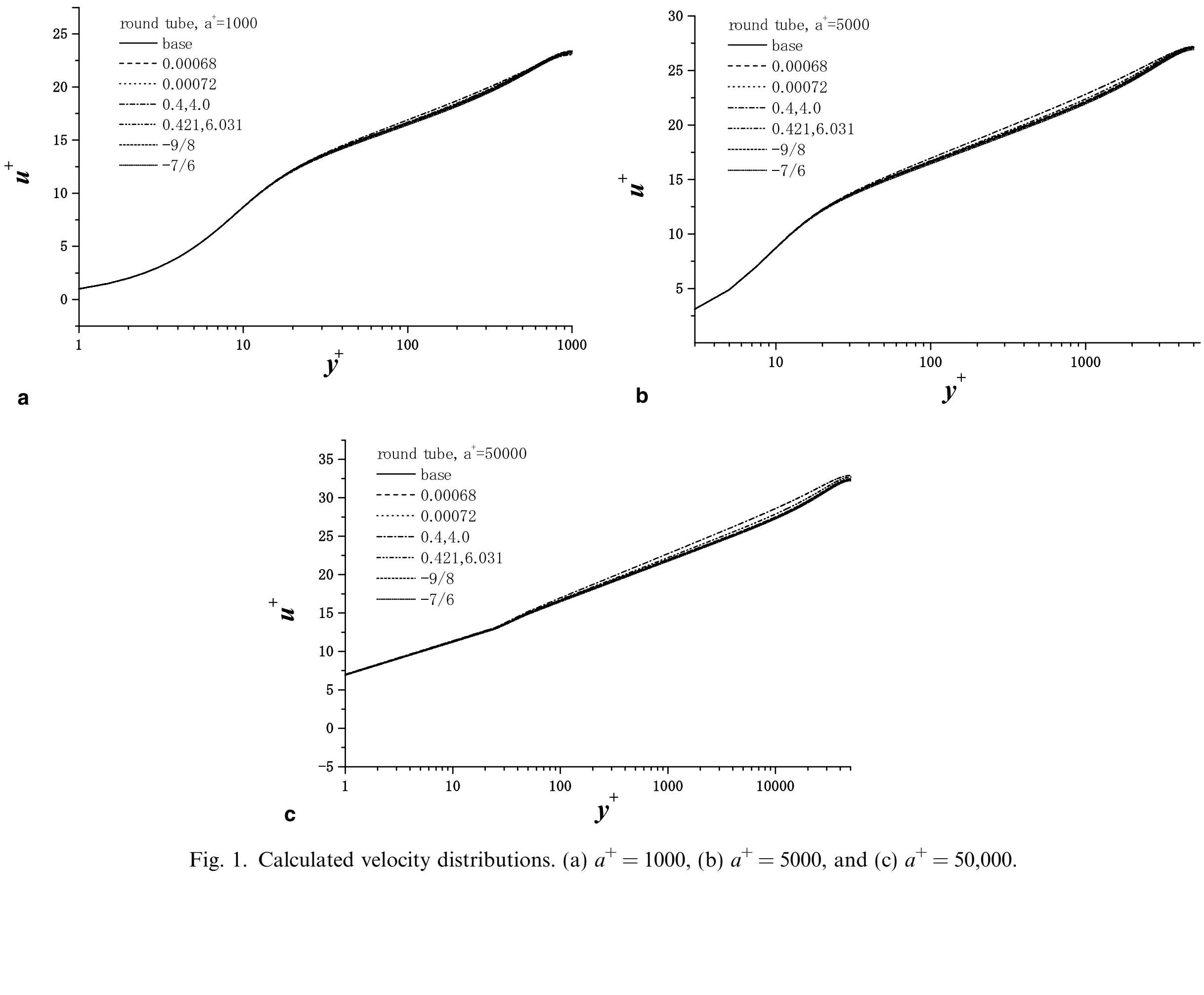

Fig. 1. Calculated velocity distributions. (a) $a^+ = 1000$, (b) $a^+ = 5000$, and (c) $a^+ = 50{,}000$.

Table 1
The sensitivity of computed values of the mixed-mean velocity in a round tube to the parameters of the algebraic model for the shear stress

α	k	A	n	u_m^+
$a^+ = 1000$ ($Re = 37{,}640$)				
0.00070	0.436	6.95	−8/7	18.82 (base case)
0.00068 (−2.8%)				18.91 (+0.48%)
0.00072 (+2.8%)				18.73 (−0.48%)
	0.400	4.00		19.05 (+1.22%)
	0.421	6.03		18.94 (+0.64%)
			−9/8 (−22%)	18.92 (+0.53%)
			−7/6 (+33%)	18.68 (−0.74%)
$a^+ = 5000$ ($Re = 226{,}900$)				
0.00070	0.436	6.95	−8/7	22.69 (base case)
0.00068 (−2.8%)				22.78 (+0.40%)
0.00072 (+2.8%)				22.59 (−0.44%)
	0.400	4.00		23.24 (+2.42%)
	0.421	6.03		22.94 (+1.10%)
			−9/8 (−22%)	22.80 (+0.48%)
			−7/6 (+33%)	22.54 (−0.66%)
$a^+ = 50{,}000$ ($Re = 2{,}801{,}000$)				
0.00070	0.436	6.95	−8/7	28.01 (base case)
0.00068 (−2.8%)				28.10 (+0.32%)
0.00072 (+2.8%)				27.91 (−0.36%)
	0.400	4.00		29.03 (+3.04%)
	0.421	6.03		28.45 (+1.57%)
			−9/8 (−22%)	28.12 (+0.39%)
			−7/6 (+33%)	27.86 (−0.43%)

perturbations of 2.8%. The resulting reverse effects on u_m^+ were approximately 0.5%, 0.4%, and 0.3% for $a^+ =$ 1000, 5000, and 50,000, respectively, which indicates almost a tenfold attenuation in uncertainty. The effect of these perturbations on $u^+$is barely distinguishable in the scale of Fig. 1(a)–(c) for any value of y^+. It may be concluded that the uncertainty of this coefficient does not affect the predictions of $u^+$and u_m^+ significantly.

3.4.2. The coefficient k and constant A

The values of $k = 0.436$ and $A = 6.95$ are those that best represent the experimental data of Zagarola [18] for u^+ as a function of y^+ and a^+ in the turbulent core. On that basis, they were chosen for Eq. (5), and thereby for the base case herein. Rather than arbitrarily perturb these values, which are interlinked, calculations were carried out for values of $k = 0.4$ and $A = 4.0$, which best represent the experimental data of Nikuradse [31], and for $k = 0.421$ and $A = 6.03$ which best represent the very recent experimental data of McKeon et al. [33], which were obtained in the same apparatus as that used by Zagarola. The values of u_m^+ calculated using the constants based on the data of Nikuradse are 1–4% higher, and those based on the data of McKeon et al. are of the order of 1% higher than those of the base case, but these differences simply characterize the differences in the three sets of measured values rather than the sensitivity of the model. The differences in u^+ are discernible in Fig. 1(a)–(c), but again primarily reflect the different sets of experimental measurements. At the same time, these differences suggest that Eq. (5) is subject to possible improvement by tweaking the values of k and A, if and when results from *DNS* are extended to higher values of Re, or better experimental data for u and/or $\overline{u'v'}$ are obtained.

3.4.3. The combining exponent n

Perturbations of −22% and +33% in the combining exponent are seen in Table 1 to be attenuated almost two orders of magnitude in the predictions of u_m^+, and in Fig. 1(a)–(c) to be indistinguishable in the predictions of u^+. This result, which might have been expected because of the well-known numerical insensitivity of the generalized correlating of Churchill and Usagi [13] to the magnitude of the combining exponent, indicates that the value of $n = -8/7$ is not a significant source of uncertainty in the predictions of u^+ and u_m^+.

3.4.4. Overall assessment

The overall conclusion from the values listed in Table 1 and plotted in Fig. 1(a)–(c) is that the values of predicted by Eqs. (2) and (3) are effectively insensitive to the numerical empiricisms of Eq. (5), and that, although improved experimental measurements of u^+ and/or

$(\overline{u'v'})^{++}$, or more precise and extended values by *DNS* may suggest numerical tweaking of the coefficients and exponents, the resulting predictions of u^+ and u^+_m are unlikely to be changed significantly. This conclusion does not extend to the exponential dependence itself, which reflects the postulated semi-logarithmic dependence of u^+, or to the combination of terms inside the absolute-value signs of Eq. (5), neither of which have been tested herein.

3.5. The sensitivity of flow in a parallel-plate channel to the parameters of the algebraic model for the shear stress

Eqs. (1)–(3) and (6) are exact for a parallel-plate channel if the dimensionless half-width of the channel, b^+, is substituted for a^+ and $Z = 1 - y/b$ for $R = 1 - y/b$. Also, Eqs. (6)–(8) are directly applicable insofar as the extended analogy of MacLeod is valid. Accordingly the computed values of u^+ are not plotted herein because they are necessarily identical to those in Fig. 1 for a round tube. However, because of the different cross-section, Eq. (4) must be replaced by

$$\left(\frac{2}{f}\right)^{1/2} \equiv u^+_m \equiv \frac{1}{b}\int_0^b u\,dy = \frac{b^+}{3}\int_0^1 [1 - (\overline{u'v'})^{++}]\,dZ^3 \tag{11}$$

The test calculations for parallel-plate channels were carried out for the same perturbations as for round tubes, but for $b^+ = 500$, 2500, and 25,000 on the presumption that the same equivalent diameter might result in almost the same values of u^+_m and *Re*. The computed values of u^+_m and the associated perturbations confirmed the numerical validity of this conjecture so closely that listing the detailed values would be redundant. All in all, the same conclusions with respect to sensitivity and accuracy as for a round tube appear to be applicable.

3.6. The sensitivity of flow in circular concentric annuli to the parameters of the algebraic model for the shear stress

Kanada et al. [26] adapted the formulation for turbulent flow in a round tube to circular concentric annuli by applying it separately to the inner and outer regions as defined by $a_1 \leqslant r \leqslant a_{max}$ and $a_{max} \leqslant r \leqslant a_2$. Here a_{max} is the radial location of the maximum in the time-averaged velocity. As contrasted with its linear variation in a round tube and in a parallel-plate channel, the radial variation of the total shear stress in a concentric circular annulus is given by a non-linear theoretical expression that incorporates a_0, the radial location of the zero in the total shear stress. It is necessary to utilize empirical correlating equations for a_{max} and a_0 as functions of a_1/a_2. The following purely empirical one of Kays and Leung [34] for a_{max} and of Rehme [35] for a_0 were used as the base case for the calculations herein:

$$\frac{a_{max} - a_1}{a_2 - a_{max}} = \left(\frac{a_1}{a_2}\right)^{0.343} \tag{12}$$

and

$$\frac{a_0 - a_1}{a_2 - a_0} = \left(\frac{a_1}{a_2}\right)^{0.386} \tag{13}$$

Because Eqs. (12) and (13) have no theoretical basis, the exact theoretical expression for $a_{max} = a_0$ in laminar flow, namely

$$\left(\frac{a_{max}}{a_1}\right)^2 = \left(\frac{a_0}{a_1}\right)^2 = \frac{[(a_2/a_1)^2 - 1]}{2\ln\{a_2/a_1\}} \tag{14}$$

was used as a test of sensitivity rather than perturbations of the empirical exponents.

The adaptation of model Eq. (5) for the inner region incorporates the constant $A = 6.95$ in the outer regions of annuli this constant is replaced by the variable value that results in matching of the two expressions for u^+ at $r = a_{max}$ for each condition.

Test calculations were carried out for values of 1000, 5000, and 50,000 for $a^+_2 - a^+_1$, the hydraulic radius of an annulus. Since u^+_m and $Re = 2(a^+_2 - a^+_1)u^+_m$ proved to be nearly the same in the limits of $a_1/a_2 = 0$, corresponding to a round tube, and $a_2/a_1 \to 1$, corresponding to a parallel-plate channel, it was speculated that these quantities might be relatively independent of a_1/a_2. Since the numerical calculations for a round tube and a parallel-plate channel were presumed to provide an adequate test of the sensitivity to the coefficients α, k, A, and n, these values were not perturbed.

The computed values in Table 2 indicate that invariance of u^+_m and Re_{4b} for a fixed value of $a^+_2 - a^+_1$ is achieved qualitatively but not quantitatively for intermediate aspect ratios. The computed values of a_{max} and a_0 are expressed in terms of $(a_{max} - a_1)/(a_2 - a_1)$ and $(a_0 - a_1)/(a_2 - a_1)$ in order to permit inclusion of values for $a_1/a_2 = 1$ and $a_1/a_2 = 0$ in the table. The near-equality for $a_1/a_2 = 0.5$ of the predictions of a_{max} and a_0 by Eqs. (12) and (13), and their small deviations from the predictions of Eq. (14), as compared to the corresponding results for $a_1/a_2 = 0.1$ may be explained by the approach to parallel-plate behavior as a_1/a_2 increases. The corresponding changes in u^+_m are difficult to explain. For example, u^+_m decreases significantly in response to only a small increase in $(a_{max} - a_1)/(a_2 - a_1)$ and $(a_0 - a_1)/(a_2 - a_1)$ for $a_1/a_2 = 0.5$, but increases somewhat less for the larger increases in $(a_{max} - a_1)/(a_2 - a_1)$ and $(a_0 - a_1)/(a_2 - a_1)$ for $a_1/a_2 = 0.1$. Such seemingly anomalous behavior suggests that the perturbation produced by the use a_{max} and a_0 for laminar flow may be too drastic to be used as a measure of sensitivity.

Table 2
The sensitivity of computed values of the mixed-mean velocity in concentric circular annuli to parameters of the algebraic model for the shear stress

	$\frac{a_{max}-a_1}{a_2-a_1}$	$\frac{a_0-a_1}{a_2-a_1}$	$(a_2^+-a_1^+)$	1000	5000	50,000
$a_1/a_2 \to 1.0$	0.5	0.5	u_m^+	18.61	22.56	27.90
			Re_{4b}	37,320	225,600	2,790,000
$a_1/a_2 = 0.5$						
Eqs. (12) and (13)	0.441	0.434	u_m^+	18.17	22.00	27.11
			Re_{4b}	36,340	220,000	2,711,000
Eq. (14)	0.471	0.471	u_m^+	16.89	20.17	25.19
			Re_{4b}	33,780	201,700	2,519,000
$a_1/a_2 = 0.1$						
Eqs. (12) and (13)	0.311	0.292	u_m^+	15.86	19.26	23.77
			Re_{4b}	31,720	192,600	2,377,000
Eq. (14)	0.404	0.404	u_m^+	16.29	20.80	26.29
			Re_{4b}	32,580	208,000	2,629,000
$a_1/a_2 \to 0$	0	0	u_m^+	18.82	22.69	28.01
			Re_{4b}	37,640	226,900	2,801,00

4. Forced convection

4.1. Basic formulations for a round tube

The thermal analogue of Eq. (1) for a round tube is

$$\frac{j}{j_w}[1-(\overline{T'v'})^{++}] = \frac{dT^+}{dy^+} \tag{15}$$

Here, $T^+ \equiv \lambda(\tau_w\rho)^{1/2}(T_w - T)/\mu j_w$ and $(\overline{T'v'})^{++} \equiv \rho c \times \overline{T'v'}/j$. The latter quantity may be recognized as the fraction of the local heat flux density at any radius that is due to the turbulent fluctuations. Since j/j_w would not be expected to differ greatly from τ/τ_w, scaling in terms of the latter is suggested. The chosen option is the replacement of j/j_w by $(1+\gamma)\tau/\tau_w$, which for a round tube becomes $(1+\gamma)R$. The quantity γ may be noted to represent the fractional deviation of the heat flux density distribution from that of the shear stress distribution. Eq. (15) is thereby converted to

$$T^+ = \frac{a^+}{2}\int_{R^2}^{1}(1+\gamma)[1-(\overline{T'v'})^{++}]\,dR^2 \tag{16}$$

Eq. (16) is more complex than Eq. (3), not only explicitly by virtue of the factor $1+\gamma$, but also implicitly by virtue of the dependence of $(\overline{T'v'})^{++}$on the Prandtl number $Pr = c\mu/\lambda$. For a round tube and uniform heating, the only thermal boundary condition to be considered herein,

$$1+\gamma = \frac{1}{R^2}\int_0^{R^2}\left(\frac{u}{u_m}\right)dR^2 \tag{17}$$

Eq. (17) reveals that γ is a function only of the velocity field and thereby of $(\overline{u'v'})^{++}$, and that it is independent of the Prandtl number. Values of γ computed from Eq. (17) by Yu et al. [36] and others confirm the expectation of restrained behavior but at the same time indicate that this quantity is not negligible with respect to unity as was postulated explicitly or implicitly in many of the early theoretical analyses of convection. The functional behavior of γ, as given by Eq. (17), has the fortuitous consequence of allowing the integration of T^+, weighted by u^+/u_m^+, over the cross-section to obtain T_m^+, to be carried out by parts, thereby resulting in the following single integral for Nu:

$$\frac{2a^+}{Nu} \equiv T_m^+ \equiv \int_0^1 T^+\left(\frac{u^+}{u_m^+}\right)dR^2 = \frac{a^+}{4}\int_0^1(1+\gamma)^2[1-(\overline{T'v})^{++}]\,dR^4 \tag{18}$$

The next step would appear to be the development of a predictive expression for $(\overline{T'v'})^{++}$ analogous to Eq. (5) for $(\overline{u'v'})^{++}$. However, it again proves to be convenient to introduce scaling in terms of the flow, namely the replacement of $1-(\overline{T'v'})^{++}$ by $\xi[1-(\overline{u'v'})^{++}]$, thereby converting Eqs. (16) and (18) to

$$T^+ = \frac{a^+}{2}\int_{R^2}^{1}(1+\gamma)\xi[1-(\overline{u'v'})^{++}]\,dR^2 \tag{19}$$

and

$$\frac{2a^+}{Nu} \equiv T_m^+ \equiv \int_0^1 T^+\left(\frac{u^+}{u_m^+}\right)dR^2 = \frac{a^+}{4}\int_0^1(1+\gamma)^2\xi[1-(\overline{u'v})^{++}]\,dR^4 \tag{20}$$

with the expectation that functional behavior of ξ is more restrained than that of $(\overline{T'v'})^{++}$.

From a comparison of Eq. (19) and its equivalent in terms of the eddy conductivity ratio, λ_t/λ, it may be shown that

$$\frac{\lambda_t}{\lambda} = \frac{(\overline{T'v'})^{++}}{1 - (\overline{T'v'})^{++}} \tag{21}$$

It follows from Eqs. (9) and (21) that

$$\xi \equiv \frac{1 - (\overline{T'v'})^{++}}{1 - (\overline{u'v'})^{++}} = \frac{(\mu + \mu_t)\lambda}{(\lambda + \lambda_t)\mu} = \frac{c(\mu + \mu_t)/(\lambda + \lambda_t)}{c\mu/\lambda} = \frac{Pr_T}{Pr} \tag{22}$$

and hence that

$$T^+ = \frac{a^+}{2} \int_{R^2}^{1} (1 + \gamma) \frac{Pr_T}{Pr} [1 - (\overline{u'v'})^{++}] \, dR^2 \tag{23}$$

and

$$\frac{2a^+}{Nu} \equiv T_m^+ = \frac{a^+}{4} \int_0^1 (1 + \gamma)^2 \frac{Pr_T}{Pr} [1 - (\overline{u'v'})^{++}] \, dR^4 \tag{24}$$

The quantity Pr_T, as defined by Eq. (22) in terms of the eddy viscosity and the eddy conductivity, is known as the total Prandtl number. However, by virtue of its representation in terms of $(\overline{u'v'})^{++}$and $(\overline{T'v'})^{++}$ it may be recognized as independent of the heuristic origins of μ_t and k_t.

Most experimental data and empirical equations for turbulent convection are expressed in terms of the turbulent Prandtl number $Pr_t \equiv c\mu_t/k$ rather than in terms of Pr_T. These values and expressions could be converted to Pr_T by means of the relationship

$$\frac{Pr}{Pr_T} = 1 + \left(\frac{Pr}{Pr_t} - 1\right)(\overline{u'v'})^{++} \tag{25}$$

but it is more convenient to re-express Eqs. (23) and (24) in terms of Pr_t. It follows from Eqs. (9) and (21) that

$$\frac{Pr_t}{Pr} \equiv \frac{c\mu_t/k_t}{c\mu/k} = \frac{(\overline{u'v'})^{++}}{(\overline{T'v'})^{++}} \left(\frac{1 - (\overline{T'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right) \tag{26}$$

Replacing $(\overline{T'v'})^{++}$ in Eqs. (16) and (18) by virtue of Eq. (26) results in

$$T^+ = \frac{a^+}{2} \int_{R^2}^{1} \frac{(1 + \gamma) \, dR^2}{1 + \frac{Pr}{Pr_t}\left(\frac{(\overline{u'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right)} \tag{27}$$

and

$$\frac{2a^+}{Nu} \equiv T_m^+ = \frac{a^+}{4} \int_0^1 \frac{(1 + \gamma)^2 \, dR^4}{1 + \frac{Pr}{Pr_t}\left(\frac{(\overline{u'v'})^{++}}{1 - (\overline{u'v'})^{++}}\right)} \tag{28}$$

Eqs. (27) and (28) appear to be more complex than Eqs. (16) and (18), but are much more convenient for numerical calculations because of the well-correlated behavior of $(\overline{u'v'})^{++}$, per Eq. (5), and the restrained variance of Pr_t/Pr. They are slightly more convenient than Eqs. (23) and (24), despite the greater variance of Pr_t relative to Pr_T in that they avoid the determination of Pr_T and display the limiting behavior for $Pr = 0$ and the special behavior for $Pr_t = Pr$ that Churchill and Zajic [37] used to devise an algebraic predictive equation for Nu. Eqs. (15)–(28) are all exact, but empiricism is invoked by the use of Eq. (5) for $(\overline{u'v'})^{++}$ and the correlating equation utilized for Pr_t or Pr_T.

Yahkot et al. [38] and Elperin et al. [39] used *renormalization group theory* to derive an algebraic theoretical expressions for Pr_T/Pr as a function of Pr and μ_T/μ, but neither of their predictions agree well with the best experimental data for some ranges of Pr and μ_T/μ, apparently as a consequence of idealizations made in the derivation. Accordingly, as discussed by Kays [40] and Churchill [41] empirical correlating equations are currently the expressions of choice for Pr_t.

Finite-difference solutions for the Nusselt number using the differential equivalents of Eqs. (3), (4) and (17), the equivalent of Eq. (17) for uniform wall-temperature, (27) and (28), together with Pr_t from the following simple expression adapted from Jischa and Rieke [42]:

$$Pr_t = 0.85 + \frac{0.15}{Pr} \tag{29}$$

were shown by Churchill and Zajic [37] to be in agreement with experimental data for all Re, all Pr, and both thermal boundary conditions within their scatter. Similar agreement was demonstrated by Danov et al. [25] for parallel-plate channels with uniform equal heating and different uniform wall temperatures, and by Yu et al. [43–45] for annuli of different aspect ratios for various combinations of uniform heating, uniform cooling , and uniform wall-temperatures. These comparisons indicate that the predictions of the new model are equal to or better than all previous ones and perhaps adequate for most practical purposes. However, in all cases, owing to the considerable scatter and the limited scope of the experimental data, these comparisons fall short of a critical test of accuracy. That shortcoming prompted the following aspect of the current work.

4.2. The sensitivity of convection in a round tube to the parameters of the algebraic model for the shear stress

Numerical calculations for T^+ and Nu were carried out for the base case using Eqs. (5) and (29), and finite-difference representations for the differential counterparts of Eqs. (3), (4), (17), (27) and (28), and then for the same parametric values as for u_m^+ in Tables 1 and 2, but additionally for a series of values of Pr and for two other expressions for Pr_t, namely a fixed value of unity, which was often postulated in early models for turbulent convection, and a semi-theoretical correlating equation of Notter and Sleicher [46] with a dependence on $(\overline{u'v'})^{++}$ as well as on Pr.

In the interests of economy and practicality, the computed values of T^+ for a round tube are not presented and the computed values of Nu are listed in Table 3 only for $Pr = 0$, which identifies the lower limiting value of

Table 3
The sensitivity of computed values of the Nusselt number in a uniformly heated round tube to parameters of the algebraic models for the turbulent shear stress and the turbulent Prandtl number

α	k	A	n	Pr_t	Pr			
					0	0.01	0.7	100
$a^+ = 1000$								
0.00070	0.436	6.95	−8/7	J&R	6.675	7.927	88.53	702.5
0.00068					6.679	7.931	88.15	696.1
0.00072					6.671	7.924	88.90	708.9
	0.400	4.00			6.727	8.036	87.73	701.2
	0.421	6.03			6.985	7.941	87.96	701.9
			−9/8		6.678	7.929	88.06	701.5
			−7/6		6.671	7.925	89.13	703.9
				N&S	6.675	7.363	89.69	719.1
				1.0	6.675	9.486	82.18	663.9
$a^+ = 5000$								
0.00070	0.436	6.95	−8/7	J&R	6.932	12.95	371.5	3480
0.00068					6.936	12.95	370.1	3449
0.00072					6.929	12.94	372.8	3511
	0.400	4.00			6.984	13.11	363.2	3470
	0.421	6.03			6.943	12.94	367.3	3476
			−9/8		6.936	12.95	369.8	3475
			−7/6		6.927	12.94	373.5	3487
				N&S	6.932	11.56	368.7	3572
				1.0	6.932	19.70	341.2	3285
$a^+ = 50{,}000$								
0.00070	0.436	6.95	−8/7	J&R	7.130	51.84	3006	34,290
0.00068					7.133	51.85	2997	33,990
0.00072					7.128	51.83	3014	34,590
	0.400	4.00			7.182	51.07	2901	34,150
	0.421	6.03			7.143	51.27	2958	34,230
			−9/8		7.134	51.85	2994	34,240
			−7/6		7.126	51.82	3020	34,360
				N&S	7.130	63.26	2969	35,090
				1.0	7.130	98.05	2732	32,320

Nu, $Pr = 0.01$, which is representative of liquid metals, $Pr = 0.7$, which corresponds to air, and $Pr = 100$, the largest value for ordinary fluids. The abbreviation J&R designates the use of Eq. (29), the slightly modified correlating equation of Jischa and Rieke [42], for Pr_t, and N&S the use of the equation of Notter and Sleicher [46].

The perturbations in the coefficients of Eq. (5) produce almost the same fractional perturbations in *Nu* for large values of *Pr* and lesser ones for small values of *Pr* that they do in u_m^+. The same correspondence was observed for T^+ and u^+, hence plots corresponding to Fig. 1 are not shown. However, the computed values of *Nu* showed a strong and what at first glance appears to be an anomalous sensitivity to the expression used for Pr_t. The values obtained from Eq. (29) are presumed to be more accurate, which is why it was chosen as the base case, but the deviations suggest the need for further measurements or theoretical calculations for this quantity (also see, Kays [40] and Churchill [41]).

Computed values of *Nu* for a parallel-plate channel and for the inner surface of concentric circular annuli heated uniformly and equally on both surfaces are listed in Tables 4 and 5, respectively. For the annuli, the term "uniform heating" designates $j_{w1}a_1 = j_{w2}a_2$ rather than $j_{w1} = j_{w2}$. The imposed perturbations in Table 4 are roughly the same as those in Table 3, except for the absence of results for the correlating equation of Notter and Sleicher. On the other hand, only a perturbation in a_0 and a_{max} is imposed in Table 5.

The fractional perturbations for a parallel-plate channel may be deduced from Table 4 to be nearly the same as those for a round tube. The most noteworthy result of Table 5 is that the equivalent-diameter concept holds very closely for a round tube and a parallel-plate channel for large values of *Pr*, holds fairly closely for $Pr = 0.7$, and fails decisively for smaller values of *Pr*.

For annuli the equivalent-diameter concept necessarily holds for $Pr \to \infty$, holds crudely for $Pr = 100$, but

Table 4
The sensitivity of computed values of the Nusselt number in a uniformly and equally heated parallel-plate channel to parameters of the algebraic models for the turbulent shear stress and the turbulent Prandtl number

α	k	A	n	Pr_t	Pr			
					0	0.01	0.7	100
$2b^+ = 1000$								
0.00070	0.436	6.95	−8/7	J&R	10.43	11.47	90.20	704.3
0.00068					10.44	11.47	89.82	697.8
0.00072					10.43	11.46	90.58	710.6
	0.400	4.00			10.52	11.64	90.86	703.6
	0.421	6.03			10.45	11.50	90.08	703.8
			−9/8		10.44	11.47	89.73	703.2
			−7/6		10.43	11.46	90.81	705.6
				1.0	10.43	12.80	83.86	665.6
$2b^+ = 5000$								
0.00070	0.436	6.95	−8/7	J&R	10.77	16.11	375.4	3484
0.00068					10.77	16.11	374.0	3452
0.00072					10.76	16.10	376.7	3515
	0.400	4.00			10.85	16.50	371.9	3477
	0.421	6.03			10.79	16.16	372.7	3480
			−9/8		10.77	16.11	373.7	3478
			−7/6		10.76	16.10	377.6	3490
				1.0	10.77	22.52	275.2	3234
$2b^+ = 50{,}000$								
0.00070	0.436	6.95	−8/7	J&R	11.01	54.43	3027	34,310
0.00068					11.01	54.44	3018	34,010
0.00072					11.00	54.42	3036	34,610
	0.400	4.00			11.09	55.00	2951	34,200
	0.421	6.03			11.03	54.25	2988	34,260
			−9/8		11.01	54.44	3015	34,260
			−7/6		11.00	54.42	3041	34,380
				1.0	11.01	109.3	2752	32,340

fails utterly for lower values, which do not vary monotonically with a_1/a_2. Calculated values of Nu for a round tube are absent from Table 5 because heating on both surfaces is a *non-sequitur*. Just as for flow, the computed values of Nu are quite sensitive to the expressions used for a_0 and a_{max}, and the trends with $a_2^+ - a_1^+$, a_1/a_2, Pr, and a_1/a_2 are inexplicable. The same conclusions as for u_m^+ are applicable for Nu.

5. Summary and conclusions

This analysis has three different aspects. The first is a description and critique of the modeling of turbulent flow and convection by means of algebraic expressions for the turbulent shear stress and the turbulent heat flux density. The second is the identification of the empiricisms and arbitrary choices in the models for these two quantities. The third is the numerical evaluation of the sensitivity of numerical predictions of turbulent flow and convection to these empiricisms and arbitrary choices. The summary and conclusions follow that division.

5.1. Summary of characteristics of the new algebraic modeling for turbulent flow

The first and most unique characteristic of the new modeling for turbulent flow is the avoidance of the introduction of a heuristic quantity such as the eddy viscosity or the mixing length. All prior models except for the κ–ε-$\overline{u'v'}$ one posit such a quantity.

A second and equally important characteristic is the choice of $(\overline{u'v'})^{++} = -\rho\overline{u'v'}/\tau$, the local fraction of the total shear stress due to the turbulence as a variable. This choice leads to simpler and more transparent formulations than any prior ones. An example is the expression of $u^+\{y^+\}$ as a simple integral, and the recognition that the double integral for $u_m^+ \equiv (2/f)^{1/2}$ can be integrated by parts resulting in a single integral for u_m^+ with an integrand identical to that for u^+. Integration by parts is possible for the formulation in terms of the

Table 5
The sensitivity of computed values of the Nusselt number for the inner surface in concentric circular annuli with uniform and equal heating on both surfaces to parameters of the algebraic model for the shear stress

$a_2^+ - a_1^+$	Re	$\frac{a_{max} - a_1}{a_2 - a_1}$	$\frac{a_0 - a_1}{a_2 - a_1}$	Pr			
				0	0.01	0.7	10
$a_1/a_2 \to 1.0$		0.5	0.5				
1000	37,220			10.43	11.47	90.20	704.3
5000	225,600			10.77	16.11	375.4	3484
50,000	2,790,000			11.01	54.43	3027	34310
$a_1/a_2 = 0.5$							
Eqs. (12) and (13)		0.441	0.434				
1000	36340			9.805	10.60	81.43	691.2
5000	220,000			9.951	14.31	344.6	3416
50,000	2,711,000			10.05	48.26	2838	33660
Eq. (14)		0.471	0.471				
1000	37,980			10.01	11.41	95.64	698.7
5000	203,700			10.20	18.44	403.9	3455
50,000	2,519,000			10.30	70.86	3280	34060
$a_1/a_2 = 0.1$							
Eqs. (12) and (13)		0.311	0.292				
1000	31,720			15.39	16.27	81.77	694.4
5000	192,600			15.44	20.63	361.8	3428
50,000	2,377,000			15.46	68.40	3191	34000
Eq. (14)		0. 404	0.404				
1000	32,580			15.25	16.23	96.26	707.8
5000	208,000			15.25	20.40	381.4	3451
50,000	2,629,000			15.25	58.22	3029	33880

eddy viscosity but was never recognized because of its greater complexity. The relationship between the eddy viscosity ratio and $(\overline{u'v'})^{++}$ in a round tube reveals that the former is a function only of the latter and therefore independent of its heuristic diffusional origin. The relationship between the mixing length and $(\overline{u'v'})^{++}$ in a round tube reveals that the former is a function only of the latter and therefore that it is independent of its heuristic origin as an analogue, but also that it is singular at the centerline, a fundamental flaw that has apparently been overlooked for 70 years.

The correlating equation that is required for $(\overline{u'v'})^{++}$ incorporates some empiricism and arbitrariness, but less than that for the κ–ε, κ–ε-$\overline{u'v'}$, and *LES* models. An inherent characteristic of the correlating equation for $(\overline{u'v'})^{++}$ is the implicit postulate of the validity of the powerful but often overlooked analogy of MacLeod and its extension to $\overline{u'v'}$ by Churchill and Chan.

The ordinary-differential formulations for u^+ and u_m^+ in terms of $(\overline{u'v'})^{++}$ are readily solved simultaneously to any required degree of accuracy by elementary finite-difference methods. This procedure is so simplistic that it may be incorporated in computational algorithms for processes such as convection or reaction.

The modeling in terms of $(\overline{u'v'})^{++}$ is directly applicable to a parallel-plate channel and is readily extended to concentric circular annuli, albeit with some additional empiricism because the locations of the maximum in the time-mean velocity and the zero in the total shear stress are not known a priori. Because these two locations are not the same, the eddy viscosity model fails for annuli but the shear-stress model remains valid.

5.2. Summary of characteristics of the new algebraic modeling for turbulent convection

The expression of the time-averaged differential energy balance in terms of $(\overline{T'v'})^{++} \equiv \rho c\overline{T'v'}/j$, which may be interpreted physically as the fraction of the total transport of energy due the turbulence, has advantages for convection equivalent to but subtly different from those for $(\overline{u'v'})^{++}$ for flow. Only the differing characteristics are noted here.

The primary differences include the dependence on the Prandtl number as a parameter, the possibility of different thermal boundary conditions, and the non-linear dependence of the local heat flux density on the distance from the wall as well as on the Prandtl number and the thermal boundary condition. Only uniform heating at the wall is considered herein but a uniform wall-temperature and, for parallel-plate channel and annuli, mixed thermal boundary conditions are readily encompassed by the algebraic modeling.

The introduction of the quantity $\gamma = (j\tau_w/j_w\tau) - 1 = (j/j_w R) - 1$ in place of the heat flux density j/j_w simplifies the modeling from a numerical point of view because γ is a perturbation. For uniform heating of a round tube or parallel-plate channel, γ is a function only of u^+/u_m^+ and thereby of $(\overline{u'v'})^{++}$, and that particular dependence fortuitously permits the integration of the double integral for Nu by parts.

A consequence of the modeling in terms of $(\overline{u'v'})^{++}$ and $(\overline{T'v'})^{++}$ is the discovery of an exact expression for the turbulent Prandtl number ratio $Pr_t/Pr \equiv (c\mu_t/\lambda_t)/(c\mu/\lambda)$ in terms of these two quantities, which demonstrates that this sometimes maligned quantity is actually well-defined in physical terms and independent of the heuristic origin of its components.

The replacement of $(\overline{T'v'})^{++}$ by Pr_t/Pr and $(\overline{u'v'})^{++}$ by virtue of this relationship results in improved differential and integral expressions for T^+ and $T_m^+ \equiv 2a^+/Nu$ by virtue of the restrained variance of Pr_t/Pr.

Expression of the integral formulations for T^+ and Nu in terms of Pr_t/Pr bestows a subtle bonus in that an asymptotic expression for $Pr \to 0$ as well as a special expression for $Pr = Pr_t$ is revealed.

An empirical relationship for Pr_t/Pr as a function of Pr and $(\overline{u'v'})^{++}$ is required for numerical calculations. The uncertainty of this relationship also affects all other forms of modeling for convection.

Just as for flow, the differential expressions for γ, T^+, and Nu can be solved readily and simultaneously by elementary finite-difference methods. Such calculations are more efficient computationally than numerical evaluation of the integral forms. Of course, T^+, Nu, and, except for uniform heating, γ are functions of Pr, and the calculations must be repeated for each value thereof. The integral forms have a supplemental value in that they provide better insight.

5.3. Summary of findings on sensitivity

The coefficient α for the third-power dependence of $(\overline{u'v'})^{++}$ on y^+ near the wall, the constant A and coefficient k for the semi-logarithmic regime of $u^+\{y^+\}$, the combining-exponent n of the power-mean of these terms are identified as principal sources of uncertainty in the correlating equation for $(\overline{u'v'})^{++}$. For concentric circular annuli the correlating equations for a_0 and a_{max} are an additional source of empiricism in the predictions for flow. For convection, the correlating expression for Pr_t is the only added source of uncertainty. The test calculations for arbitrary perturbations of the coefficient α and the combining-exponent indicate that the predictions of u^+, u_m^+, and Nu are very insensitive to the perturbations, and that the probable uncertainty in these quantities does not effect the accuracy of the predictions significantly.

Rather than arbitrarily perturbing the von Kármán coefficient k of the semi-logarithmic representation for the time-averaged velocity "in the turbulent core near the wall" and the constant A, which is a measure of the magnitude of the wake, the predictions of u^+, u_m^+, and Nu were compared for the values of k and A as determined from three different sets of experiments. The predictions of u_m^+ and Nu differed in the range of 1–3%. This is effectively a measure of the difference in the experimental determinations, but it leads to two conclusions regarding the coefficients, first, a sensitivity of that order of magnitude, and second, the need for definitive experiments or numerical solutions by *DNS* or the equivalent. An alternative conclusion is that a better model needs to be formulated for this regime.

The sensitivity to the empirical expression used for a_0, a_{max}, and Pr_t was also examined by comparing the predictions for alternative choices of these expressions. In the case of a_0 and a_{max} the arbitrary alternative was the theoretical solution for laminar flow. The rather significant sensitivity suggests that efforts should be made to obtain improved experimental data and/or correlating equations for all three of these quantities.

5.4. Overall conclusions

Algebraic modeling of turbulent flow in terms of the shear stress and of turbulent convection in terms of the turbulent heat flux density is shown to be simpler than modeling in terms of the eddy diffusivity and the eddy conductivity. As a supplement to prior comparisons of the predicted values with scattered experimental data the predictions of u^+, u_m^+, and Nu are shown to be insensitive to most of the empirical coefficients and exponents of the correlating equations for the turbulent shear stress $(\overline{u'v'})^{++}$. The principal sources of uncertainty in the predictions appear to be those associated with the empirical expressions for the time-averaged velocity in "the turbulent core near the wall", the turbulent Prandtl number, and the locations of the maximum in the time-averaged velocity and the zero in the total shear stress. Further experimental work or extension of *SDS* to resolve these particular uncertainties appears to be worthwhile. Pending such improvements, the combination of Eqs. (2), (4), and (5) for flow in round tubes; the combination of Eq. (11) with Eqs. (2) and (5) in terms of b^+ for flow in parallel-plate channels; and the complete model described by Kanada et al. [26] for flow in annuli, appear to be more reliable than any alternatives. Likewise, the combination of Eqs. (5), (17), (28), and (29) for convection for convection in a round tube is equally superior, as are the equivalents for parallel-plate channels and annuli with all combinations of uniform heating and uniform wall-temperatures, as described by Yu et al. [43–45].

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